Everyday relativity and the Doppler effect

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It is generally believed that special relativistic effects are important only when studying objects moving at speeds close to that of light. This belief leaves many practicing scientists and engineers with the impression that an understanding of relativity is not necessary for their day jobs. Our aim is to show that the ideas and mathematics of the special theory of relativity are used in practical applications involving objects moving much slower than the speed of light. In particular, we show how the Doppler shift for sound and light can be calculated from the postulates of relativity.

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I. INTRODUCTION

The Doppler effect is the phenomenon that there is a difference between the received frequency f_r of a wave and the emitted frequency f_e of the wave when the emitter and/or the receiver is moving. There are many applications of the Doppler effect including radar,[1] global navigation satellite systems,[2] and medical imaging.[3]

Traditionally, the Doppler formula is derived by calculating the wavelength measured by the receiver if either the receiver or emitter is moving, Fig. 1 shows the wave fronts of a moving emitter. Given the wavelength λ and the phase speed u_p, the frequency is calculated from $f = u_p/\lambda$ (see, for example, Sec. 15–5 of Ref. 4). An alternate approach is to use coordinate transformations to change from a frame of reference (coordinate system) in which the emitter is at rest to one in which the receiver is at rest, and calculate the corresponding received frequency. This second approach is the way in which the relativistic Doppler effect for light is calculated (see, for example, Sec. 10.2 of Ref. 5). Some authors have shown how to apply the Lorentz transformations to acoustic signals to obtain the generalized relativistic Doppler effect.[6–9] Our paper builds on their work.

In this paper, we use the first postulate of relativity along with the Galilean transformations to derive the nonrelativistic Doppler effect formula. This approach not only gets students thinking in a relative way but also gets them used to space-time transformations without being bothered by the counterintuitive properties of the Lorentz transformations. Furthermore, by using the Galilean relativity approach to derive the Doppler formula, we can easily show how it is modified by a moving medium—wind in the case of sound—by transforming to an observer that is stationary with respect to the medium.

The Doppler effect derived from the Galilean transformations is inconsistent with the principle that the Doppler effect for light is the same if the emitter is moving towards the receiver or the receiver is moving towards the emitter. We show how the Lorentz transformations resolve this contradiction and, as a direct consequence, predict angular aberration. The resulting formula is equally applicable to both light and sound.

II. THE FIRST POSTULATE OF SPECIAL RELATIVITY AND ITS APPLICATION TO WAVES

The first postulate of special relativity is:[10] "The laws of physics are the same to all inertial observers." An inertial observer is one for whom Newton's first law holds: "If there is no net force on an object then it will not accelerate."

Consider the two observers $\mathcal{O}$ and $\mathcal{O}'$, with $\mathcal{O}'$ moving at a constant velocity $\boldsymbol{v}$ with respect to $\mathcal{O}$, as depicted in Fig. 2. Each observer sets up a coordinate system to record the times and locations of events. Relativity theory is about understanding how the locations and times of events in one coordinate system are related to those in another. While the two observers might disagree on *when* and *where* an event happened, they always agree on *what* happened.

A. Galilean transformations

Prior to the development of the special theory of relativity, it was assumed that if the $\mathcal{O}$ and $\mathcal{O}'$ coordinate axes are aligned at $t' = t = 0$, then the location of an event in $\mathcal{O}'$ is related to the location in $\mathcal{O}$ by

$$\boldsymbol{r} = \boldsymbol{r}' + \boldsymbol{v}t'. \tag{1}$$

Furthermore, it was assumed that the time that an event occurred was agreed upon by all observers (assuming that their clocks are synchronised) so that

$$t = t'. \tag{2}$$

Equations (1) and (2) are known as the Galilean transformations and can be expressed as a single matrix equation

$$\boldsymbol{\xi} = \boldsymbol{\Lambda}_G \boldsymbol{\xi}', \tag{3}$$

where $\boldsymbol{\xi}$ and $\boldsymbol{\xi}'$ are space-time vectors

$$\boldsymbol{\xi} \equiv \begin{bmatrix} t \\ \boldsymbol{r} \end{bmatrix}, \quad \boldsymbol{\xi}' \equiv \begin{bmatrix} t' \\ \boldsymbol{r}' \end{bmatrix}, \tag{4}$$

and $\boldsymbol{\Lambda}_G$ is the Galilean transformation matrix

$$\boldsymbol{\Lambda}_G \equiv \begin{bmatrix} 1 & 0 & 0 & 0 \\ v^x & 1 & 0 & 0 \\ v^y & 0 & 1 & 0 \\ v^z & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \boldsymbol{0}^T \\ \boldsymbol{v} & \boldsymbol{I}_3 \end{bmatrix}. \tag{5}$$

Here $\boldsymbol{0}$ is a 3×1 vector of zeros and $\boldsymbol{I}_3$ is the 3×3 identity matrix.

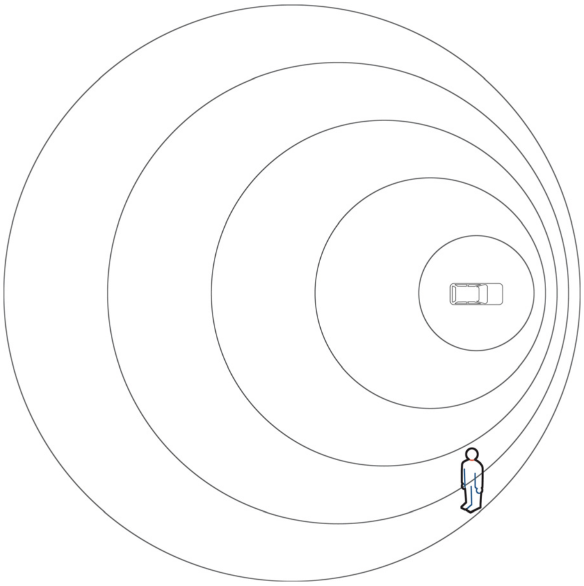

Fig. 1. The moving car is emitting sounds that are received by a person on the side of the road. Depending on where the person is standing, the frequency may be increased or decreased.

B. The first postulate and phase invariance

The magnitude of a monotone plane wave at a point $\boldsymbol{r}$ at time t is

$$A(\boldsymbol{r},t) = A_0 \cos[\varphi(\boldsymbol{r},t)] = A_0 \cos(\boldsymbol{k}^T \boldsymbol{r} - 2\pi f t), \tag{6}$$

where the constants A_0, $\boldsymbol{k}$, and f are the amplitude, wave vector, and frequency.[11]

One of the first things that students of relativity need to get used to doing is writing equations in terms of space-time vectors. For example, Eq. (6) can be written as

$$A(\boldsymbol{r},t) = A_0 \cos(\boldsymbol{\kappa}^T \xi), \tag{7}$$

where $\boldsymbol{\kappa}$ is the *wave four-vector*

$$\boldsymbol{\kappa} \equiv \begin{bmatrix} -2\pi f \\ \boldsymbol{k} \end{bmatrix}, \tag{8}$$

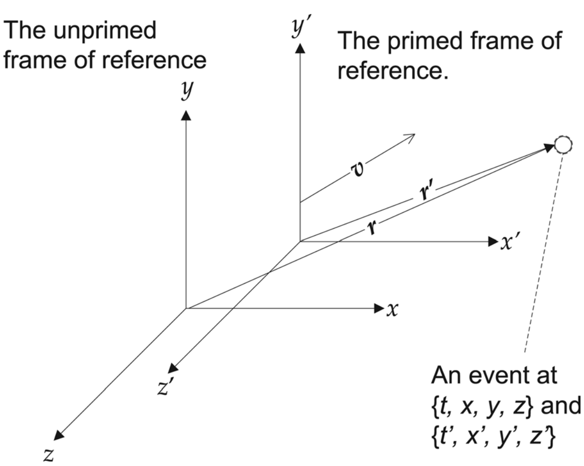

Fig. 2. An event in two space-time coordinate systems. The primed coordinated system is moving with a velocity $\boldsymbol{v}$ with respect to the unprimed system.

and ξ is the space-time vector given in Eq. (4).

The Galilean transformations, Eq. (3), tell us how to compare the space-time coordinates (ξ and ξ') of events in two inertial frames. But how does the wave four-vector $\boldsymbol{\kappa}$ transform? To use relativity theory we need to understand which quantities depend on the relative velocity between two observers and which do not. For example, as we are about to show, while the space-time coordinates ξ of points on a wave and the wave four-vector $\boldsymbol{\kappa}$ depend on the velocity of an observer, the phase $\varphi = \boldsymbol{\kappa}^T \xi$ of the wave does not.

Consider the events depicted in Fig. 3:

Event 1: The Mth wave front passes through the person.

Event 2: The $(M+N)$th wave front passes through the person.

According to $\mathcal{O}$, the time between these two events is Δt and, as the person hasn't moved, they occur at the same place; therefore, $\Delta r = 0$ so that $\Delta\xi^T = [\Delta t, 0, 0, 0]$. On the other hand, according to $\mathcal{O}'$ the person has moved $\Delta \boldsymbol{r}' = -\boldsymbol{v}\Delta t$, but the time between the two events is still Δt so that $\Delta\boldsymbol{\xi}'^T = [\Delta t, -\boldsymbol{v}^T \Delta t]$.

Although the two observers do not agree on where the second event occurred, they agree on what occurred—N wave fronts passed through the person. The number of wave fronts counted by an observer is simply the total phase shift measured between events 1 and 2 divided by 2π:

$$N = \frac{\Delta\varphi}{2\pi} = \frac{\boldsymbol{k}^T \Delta\xi}{2\pi} = \frac{\Delta\varphi}{2\pi} = \frac{\boldsymbol{k}'^T \Delta\xi'}{2\pi} \Rightarrow \boldsymbol{\kappa}^T \xi = \boldsymbol{\kappa}'^T \xi'. \tag{9}$$

By applying the Galilean transformations [Eqs. (3)–(5)] to the principle of phase invariance Eq. (9) one finds that the transformation equation for the wave four-vector is

$$\boldsymbol{\kappa}' = \boldsymbol{\Lambda}_G^T \boldsymbol{\kappa}. \tag{10}$$

The first row of Eq. (10) tells us how the frequency measured by $\mathcal{O}'$ can be determined from the wave parameters f and $\boldsymbol{k}$ measured in the $\mathcal{O}$ frame:

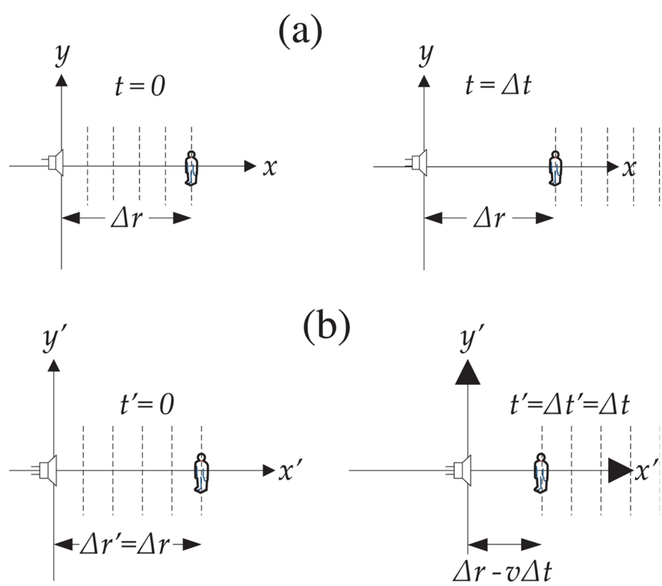

Fig. 3. The number of wave fronts (dashed lines) as determined by (top) an inertial frame $\mathcal{O}$ in which both the receiver and the emitter are at rest, and (bottom) an inertial frame $\mathcal{O}'$ in which the receiver is moving with a velocity $-\boldsymbol{v}$.

$$f' = f - \frac{\boldsymbol{v}^T \boldsymbol{k}}{2\pi} = f - \frac{\boldsymbol{k}^T \boldsymbol{v}}{2\pi}. \tag{11}$$

The second through fourth rows of Eq. (10) establish that the wave vector $\boldsymbol{k}$ is the same to both observers:

$$\boldsymbol{k}' = \boldsymbol{k}. \tag{12}$$

C. Why don't the relativistic and standard approaches to calculating the Doppler effect agree?

The invariance of $\boldsymbol{k} \equiv 2\pi\hat{\boldsymbol{k}}/\lambda$ derived in the previous section predicts that all inertial observers agree on both the **wavelength** and **direction** of the wave. This invariance of $\boldsymbol{k}$ leads to two apparent contradictions between the standard undergraduate textbook approach to the Doppler effect (such as can be found in Ref. 4 for example) and the derivation given here. The apparent contradictions are:

1. The standard approach predicts that the wavelength measured by the receiver differs from the emitted wavelength if the emitter is moving.
2. The aberration effect predicts that the direction of a wave depends on the relative motion between the emitter and the receiver.

1. *The apparent wavelength contradiction*

In many undergraduate physics textbooks the Doppler effect for a moving emitter is calculated by determining the shortening or lengthening of wavelengths due to the motion of the emitter.[4,12] In the simple case of an emitter moving directly towards a stationary receiver, as depicted in Fig. 4, the received wavelength is

$$\lambda_r = \lambda - \frac{v_e}{f_e}, \tag{13}$$

where λ_r is the wavelength as determined by the receiver, f_e is the frequency of the signal as determined by the emitter, v_e is the speed of the emitter, and λ is the "normal" value of the wavelength (the one everyone would agree on if there was no relative motion).

The difference between the normal wavelength λ and the received wavelength λ_r appears to be in direct contradiction with Eq. (12), which implies that both the emitter and the receiver record the same wavelength. This apparent contradiction is resolved when one realizes that λ in Eq. (13) is *not* the wavelength in the rest frame of the emitter but rather the distance traveled by the wave-front in one wave period $T_e = 1/f_e$ as measured in the receiver's rest frame; i.e., $\lambda = u_p/f_e$, where u_p is the phase speed of the wave as measured by the receiver (in the rest frame of the medium).

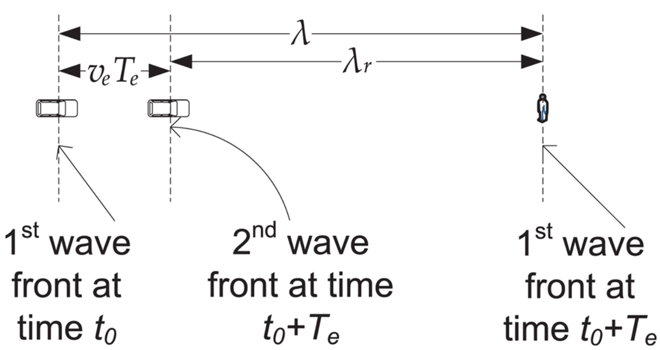

Fig. 4. The apparent difference between the received and emitted wavelength due to the motion of the emitter.

In the rest frame of the emitter, the phase speed of the wave u_p^e is reduced by the speed of the emitter: $u_p^e = u_p - v_e$. The wavelength in the emitter's rest frame can then be calculated as

$$\lambda_e = \frac{u_p^e}{f_e} = \frac{u_p - v_e}{f_e} = \lambda - \frac{v_e}{f_e}, \tag{14}$$

and from Eq. (13) we see that $\lambda_e = \lambda_r$ so that the emitter and the receiver do indeed measure the same wavelength.

2. *The apparent aberration contradiction*

As we will show in Sec. II D, the direction of the phase velocity $\boldsymbol{u}_p$ is the same as the direction of the wave vector $\boldsymbol{k}$ and hence from Eq. (12) is the same in all inertial frames. On the other hand, both the speed and direction of the group velocity $\boldsymbol{u}_g$ depend on the velocities of the receiver, emitter, and medium (for waves that travel in a medium). Aberration is a property of the group velocity, not the phase velocity, and hence there is no contradiction between the aberration effect and Eq. (12). A derivation of non-relativistic aberration is given in appendix A.

D. Galilean transformations of the wave and group velocities

The group velocity and phase velocity transform differently under Galilean transformations. The group velocity of a wave is[13,14]

$$\boldsymbol{u}_g = 2\pi \frac{\partial f}{\partial \boldsymbol{k}^T}, \tag{15}$$

and from the Doppler effect formulas (11) and (12), we can determine how the group velocity changes between inertial observers:

$$\boldsymbol{u}_g' = 2\pi \frac{\partial f'}{\partial \boldsymbol{k}'^T} = 2\pi \frac{\partial f}{\partial \boldsymbol{k}^T} - \frac{\partial \boldsymbol{k}^T \boldsymbol{v}}{\partial \boldsymbol{k}^T} \Rightarrow \boldsymbol{u}_g' = \boldsymbol{u}_g - \boldsymbol{v}. \tag{16}$$

From this, we can conclude that the measured values of both the magnitude and direction of the group velocity depend on the velocity of the observer.

The phase velocity $\boldsymbol{u}_p$ is defined as the rate of change of $\boldsymbol{r}$ in the direction of $\boldsymbol{k}$ that keeps the phase constant, i.e., $A(\boldsymbol{r} + \boldsymbol{u}_p \Delta t, t + \Delta t) = A(\boldsymbol{r}, t)$. For a plane wave given by Eq. (6), we have that

$$\boldsymbol{k}^T \boldsymbol{u}_p \Delta t - 2\pi f \Delta t = 0 \Rightarrow \boldsymbol{k}^T \boldsymbol{u}_p - 2\pi f = 0. \tag{17}$$

As $\boldsymbol{k}$ is parallel to the phase velocity $\boldsymbol{u}_p$, it follows from Eq. (17) that

$$\boldsymbol{k} = \frac{2\pi f}{u_p^2} \boldsymbol{u}_p. \tag{18}$$

The phase velocity equation ($\boldsymbol{k}^T \boldsymbol{u}_p = 2\pi f$) along with the transformation laws for f and $\boldsymbol{k}$ [Eqs. (11) and (12)] can be used to determine the transformation law for the phase velocity

$$\begin{aligned}
\boldsymbol{k}'^T \boldsymbol{u}'_p &= 2\pi f'; \\
\boldsymbol{k}^T \boldsymbol{u}'_p &= 2\pi\left(f - \frac{\boldsymbol{v}^T\boldsymbol{k}}{2\pi}\right) \quad \text{from Eq. (11)} \\
&= 2\pi f\left(1 - \frac{\boldsymbol{v}^T\boldsymbol{u}_p}{u_p^2}\right) \quad \text{from Eq. (18)} \\
&= \boldsymbol{k}^T\boldsymbol{u}_p\left(1 - \frac{\boldsymbol{v}^T\boldsymbol{u}_p}{u_p^2}\right) \quad \text{from Eq. (17)} \\
&\Rightarrow \boldsymbol{u}'_p = \boldsymbol{u}_p\left(1 - \frac{\boldsymbol{v}^T\boldsymbol{u}_p}{u_p^2}\right). \qquad (19)
\end{aligned}$$

From this, we conclude that while the magnitude of the measured phase velocity depends on the velocity of the observer, the direction does not.

III. THE DOPPLER EFFECT FORMULA

Using Eq. (18), we can express the Doppler formula (11) in terms of the phase velocity:

$$f' = f\left(1 - \frac{\boldsymbol{v}^T\boldsymbol{u}_p}{u_p^2}\right). \qquad (20)$$

A. The Doppler effect for waves traveling in a medium

For waves (such as sound waves) that travel in a medium, the preferred frame of reference is the rest frame of the medium. In this frame the phase velocity is known and it is identical to the group velocity for (monotone) plane waves. The phase velocity in the rest frame of the emitter is determined by Eq. (19) to be

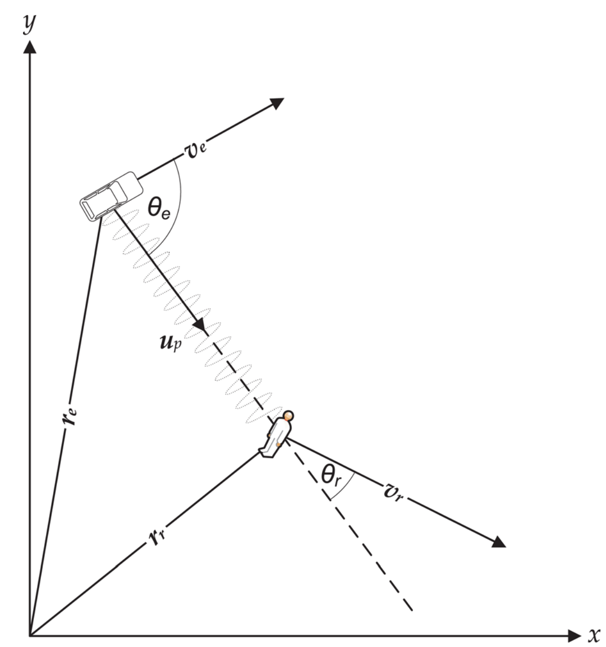

Fig. 5. A receiver moving with a velocity v_r receives a signal from an emitter moving with a velocity v_e.

$$\boldsymbol{u}_e = \boldsymbol{u}_p\left(1 - \frac{\boldsymbol{v}_e^T\boldsymbol{u}_p}{u_p^2}\right), \qquad (21)$$

where $\boldsymbol{u}_p$ is the phase velocity of the wave in the rest frame of the medium.

The received frequency is calculated using Eq. (20):

$$f_r = f_e\left(1 - \frac{(\boldsymbol{v}_r^T - \boldsymbol{v}_e^T)\boldsymbol{u}_e}{u_e^2}\right) = f_e\left(\frac{u_p^2 - \boldsymbol{v}_r^T\boldsymbol{u}_p}{u_p^2 - \boldsymbol{v}_e^T\boldsymbol{u}_p}\right). \qquad (22a)$$

It is more common to see the Doppler effect equation in trigonometric form

$$f_r = \frac{1 - (v_r/u_p)\cos\theta_r}{1 - (v_e/u_p)\cos\theta_e} f_e, \qquad (22b)$$

where θ_r and θ_e are the angles between the phase velocity vector and the receiver's and emitter's velocity vectors, respectively (see Fig. 5). Notice that if $\theta_r < \pi/2$ the receiver

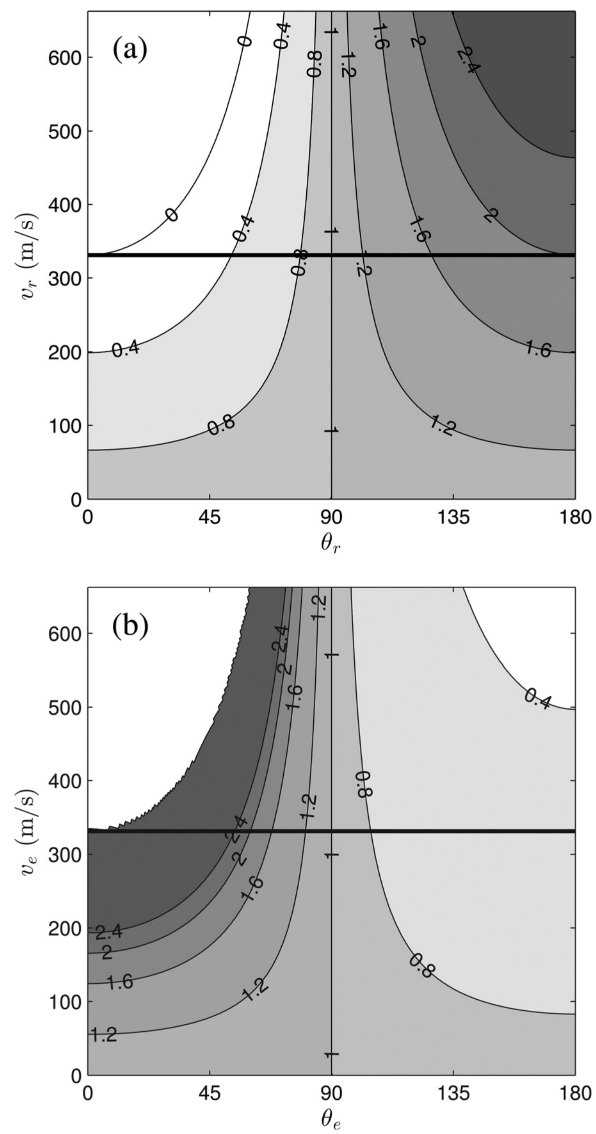

Fig. 6. Contour plots of the ratio of received to emitted frequencies f_r/f_e for the Doppler effect in the rest frame of the emitter for: (a) moving receiver and stationary emitter, and (b) moving emitter and stationary receiver. The solid horizontal line is the speed of sound in air u.

is moving away from the emitter but if $\theta_e < \pi/2$ the emitter is moving towards the receiver.

Figure 6 shows the contour plots for the Doppler shift over a range of speeds and angles. The white region in Fig. 6(a) indicates that the receiver is moving away from the emitter faster than the wave and hence there is no received signal. The white region in Fig. 6(b) indicates that the emitter is moving towards the receiver faster than the phase speed. Notice that as the emitter approaches the speed of sound a wall of sound is generated that has an infinite frequency; this is known as a *sonic boom*.

B. Wind and the Doppler effect

One of the advantages of the relativistic approach to deriving the Doppler effect is that it makes it easier to generalize to the case when the medium itself is moving, such as for sound waves traveling in wind. To calculate the Doppler effect in this case, one puts oneself in the frame that is at rest with respect to the medium (which for sound waves in air is a frame of reference that is co-moving with the wind).

In the rest frame of the medium the emitter has a velocity $\boldsymbol{v}_e - \boldsymbol{v}_m$, where $\boldsymbol{v}_m$ is the velocity of the medium. From Eq. (20), we have

$$f_e = f_m\left[1 - \frac{(\boldsymbol{v}_e^T - \boldsymbol{v}_m^T)\boldsymbol{u}_p}{u_p^2}\right], \tag{23}$$

where f_m is the frequency in the rest frame of the medium. Similarly, the received frequency f_r can be determined in terms of the rest frame of the medium:

$$f_r = f_m\left[1 - \frac{(\boldsymbol{v}_r^T - \boldsymbol{v}_m^T)\boldsymbol{u}_p}{u_p^2}\right]. \tag{24}$$

Eliminating f_m from Eqs. (23) and (24), we find that

$$f_r = f_e\left(\frac{u_p^2 - \boldsymbol{v}_r^T\boldsymbol{u}_p + \boldsymbol{v}_m^T\boldsymbol{u}_p}{u_p^2 - \boldsymbol{v}_e^T\boldsymbol{u}_p + \boldsymbol{v}_m^T\boldsymbol{u}_p}\right) \tag{25a}$$

$$= f_e\left[\frac{1 - (v_r/u)\cos\theta_r + (v_m/u)\cos\theta_m}{1 - (v_e/u)\cos\theta_e + (v_m/u)\cos\theta_m}\right], \tag{25b}$$

where v_m is the speed of the wind and θ_m is the angle between the wind vector and the phase velocity vector.

If the speeds of the emitter, transmitter, and wind are much less than the phase speed of the wave then the Taylor expansion of Eq. (25) gives

$$f_r \approx f_e\left[1 - \frac{(\boldsymbol{v}_r - \boldsymbol{v}_e)^T\boldsymbol{u}_p}{u_p^2}\right], \tag{26}$$

which means that to first order the wind doesn't alter the magnitude or direction of the Doppler effect.

Comparing Figs. 6(a) and 7, we see that if the wind speed is comparable to the speed of sound then the wind can have a significant effect on the received frequency.

IV. THE PROBLEM OF LIGHT

Unlike sound, light doesn't travel in a medium, and therefore there is no way of telling whether the receiver or the emitter is moving; all that is known is that there is a relative velocity between the two. The equivalence between a receiver moving toward a light source and a light source moving toward a receiver is inconsistent with the Doppler formula (22), which gives a different result depending on whether the emitter or the receiver is moving.

One of the great insights to come from relativity theory was the realization that the Galilean transformations are wrong. The correct way to translate the space-time measurements of events between inertial frames is with the Lorentz transformations.

A. Lorentz transformations

The Lorentz transformations can be found in many textbooks (see, for example, Ref. 16, [p. 29] or Ref. 15, [p. 280]) and in Cartesian coordinates they are

$$t = \gamma t' + \gamma \boldsymbol{v}^T\boldsymbol{r}'/c^2 \tag{27a}$$

$$\boldsymbol{r} = \gamma\boldsymbol{v}t' + \left(\boldsymbol{I}_3 + \frac{\gamma - 1}{v^2}\boldsymbol{v}\boldsymbol{v}^T\right)\boldsymbol{r}'. \tag{27b}$$

These equations can be written in linear algebra form as

$$\boldsymbol{\xi} \equiv \boldsymbol{\Lambda}_L\boldsymbol{\xi}', \tag{28}$$

where $\boldsymbol{\xi}^T = [t, x, y, z]$ and $\boldsymbol{\xi}'^T = [t', x', y', z']$ are the space-time coordinates, and $\boldsymbol{\Lambda}_L$ is the Lorentz transformation matrix

$$\Lambda_L = \begin{bmatrix} \gamma & \gamma v^x/c^2 & \gamma v^y/c^2 & \gamma v^z/c^2 \\ \gamma v^x & 1 + \dfrac{(v^x)^2(\gamma-1)}{v^2} & v^x v^y\dfrac{(\gamma-1)}{v^2} & v^x v^z\dfrac{(\gamma-1)}{v^2} \\ \gamma v^y & v^x v^y\dfrac{(\gamma-1)}{v^2} & 1 + \dfrac{(v^y)^2(\gamma-1)}{v^2} & v^y v^z\dfrac{(\gamma-1)}{v^2} \\ \gamma v^z & v^x v^z\dfrac{(\gamma-1)}{v^2} & v^y v^z\dfrac{(\gamma-1)}{v^2} & 1 + \dfrac{(v^z)^2(\gamma-1)}{v^2} \end{bmatrix} \tag{29a}$$

$$= \begin{bmatrix} \gamma & \gamma\boldsymbol{v}^T/c^2 \\ \gamma\boldsymbol{v} & \boldsymbol{I}_3 + \dfrac{\gamma-1}{v^2}\boldsymbol{v}\boldsymbol{v}^T \end{bmatrix}. \tag{29b}$$

As usual, γ is the Lorentz factor

$$\gamma \equiv \frac{1}{\sqrt{1 - v^2/c^2}} \tag{30}$$

and $v = \sqrt{\boldsymbol{v}^T \boldsymbol{v}}$ is the speed of $\mathcal{O}'$ relative to $\mathcal{O}$. Notice that in the limit $\boldsymbol{v}/c \to 0$ the Lorentz factor $\gamma \to 1$ and the Lorentz transformations Eq. (29) are the same as the Galilean transformations Eq. (5).

Applying the Lorentz transformations to the principle of phase invariance ($\boldsymbol{\kappa}'^T \boldsymbol{\xi}' = \boldsymbol{\kappa}^T \boldsymbol{\xi}$), we find that

$$f' = \gamma \left(1 - \frac{\boldsymbol{v}^T \boldsymbol{u}_p}{u_p^2} \right) f. \tag{31}$$

As shown in Appendix B, the Doppler effect derived using the Lorentz transformations gives

$$f_r = \frac{\sqrt{1 - v_e^2/c^2}}{\sqrt{1 - v_r^2/c^2}} \frac{\left(u_p^2 - \boldsymbol{v}_r^T \boldsymbol{u}_p\right)}{\left(u_p^2 - \boldsymbol{v}_e^T \boldsymbol{u}_p\right)} f_e \tag{32a}$$

$$= \frac{\sqrt{1 - v_e^2/c^2}}{\sqrt{1 - v_r^2/c^2}} \frac{\left[1 - (v_r/u_p)\cos\theta_r\right]}{\left[1 - (v_e/u)\cos\theta_e\right]} f_e. \tag{32b}$$

The special-relativistic Doppler formula given by Eq. (32) is applicable to all situations. The Doppler effect for a moving emitter is shown in Fig. 8. Equation (32) is equally valid for light and sound, although for sound, $\boldsymbol{v}_r$, $\boldsymbol{v}_e$, and $\boldsymbol{u}_p$ are velocities relative to the rest frame of the medium. In the case that the emitter and receiver velocities are much less than the speed of light ($v/c \to 0$) the Lorentz transformations reduce to the Galilean ones and the generalized Doppler shift formula Eq. (32) returns the non-relativistic result given by Eq. (22).

As discussed in Ref. 5 (p. 204), for light the phase speed is $u_p = c$ and the frequency in the rest frame of the emitter (defined by $\boldsymbol{v}_e = 0$) is known, so that Eq. (32) becomes

$$f_r = \frac{1 - (v_r/c)\cos\theta_r}{\sqrt{1 - v_r^2/c^2}} f_e. \tag{33}$$

Equivalently, as discussed in Ref. 10 (p. 143), in the rest frame of the receiver ($\boldsymbol{v}_r = 0$) the emitter is moving with a velocity $\boldsymbol{v}_e$ and from Eq. (32) the measured frequency is

$$f_r = \frac{\sqrt{1 - v_e^2/c^2}}{1 - (v_e/c)\cos\theta_e} f_e. \tag{34}$$

B. Relativistic aberration

From the first postulate of special relativity, it is impossible for a receiver in a vacuum to determine if the receiver is moving towards the emitter or vice versa. A direct consequence of this is that the Doppler shift for a receiver moving with a velocity $\boldsymbol{v}_r$ toward a stationary emitter must be the same as for an emitter moving with a velocity $\boldsymbol{v}_e = -\boldsymbol{v}_r$ toward a stationary receiver. This requirement is equivalent to the requirement that Eqs. (33) and (34) are the same if $v_r = v_e$ and $\theta_r = \theta_e + \pi$; however, this is not the case!

This apparent contradiction is resolved when one realizes that the angle between vectors is not preserved under the

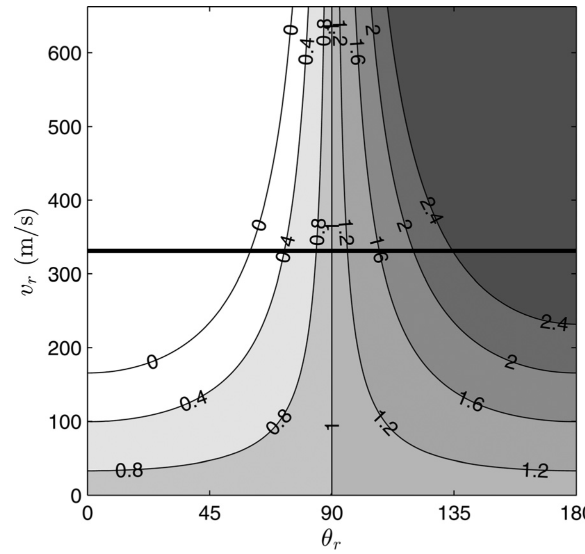

Fig. 7. Contour plot of f_r/f_e for the Doppler effect with wind for a moving receiver and stationary emitter. The speed of the wind is half the speed of sound and its direction is 45° to the phase velocity.

Lorentz transformations described by Eq. (27b). Substituting $v_r = v_e$ and equating Eq. (33) with Eq. (34), we find that

$$\cos\theta_r = \frac{(v_e/c) - \cos\theta_e}{1 - (v_e/c)\cos\theta_e}. \tag{35}$$

This effect is the well known *relativistic aberration* effect and it is normally derived directly from the Lorentz transformations (see Ref. 10, p. 133). However, as we have just shown, it is a direct consequence of applying the first postulate of relativity to the Doppler effect for light.

V. CONCLUSION

The first postulate of relativity essentially states that the laws of physics don't depend on the velocity of the observer. We have shown how to use this principle to derive the Doppler effect formula. Our approach can help

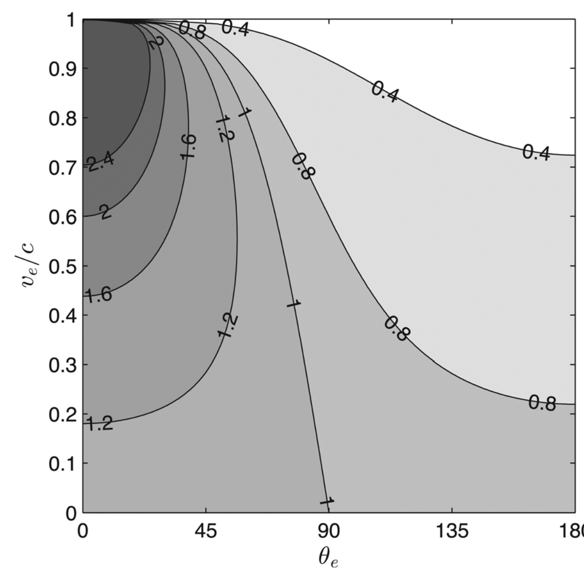

Fig. 8. Contour plot of f_r/f_e for the relativistic Doppler effect for a fast moving light source.

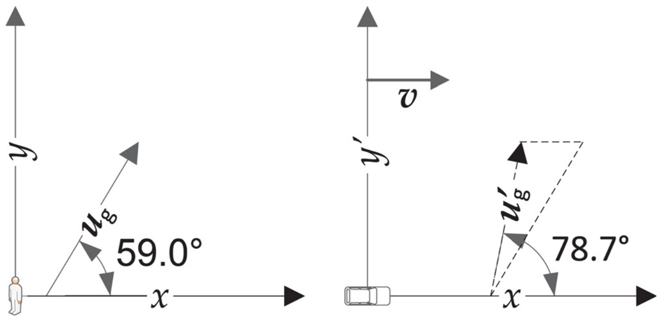

Fig. 9. Aberration of the group velocity due to motion of the receiver. The observer $\mathcal{O}'$ is moving with a velocity $\boldsymbol{v}$ with respect to $\mathcal{O}$.

undergraduate students understand the principles and applications of special and Galilean relativity theory before being exposed to the more counter-intuitive results of special relativity, such as time dilation and length contraction.

By considering Galilean transformations we have shown how to derive the Doppler effect in the presence of wind [Eq. (25)], which is not commonly done in undergraduate textbooks. In so doing we demonstrated that if the wind speed is much less than the wave speed then the wind has no effect, while if the wind speed is comparable to the wave speed then it has a significant effect on the received frequency, as seen by Fig. 7.

The Doppler effect derived using the Galilean transformations predicts different frequency shifts depending on whether the emitter or the receiver is moving. This result is inconsistent with the observation that, for light in a vacuum, only the relative velocity between the emitter and receiver is important. We showed that the requirement that the Doppler effect for light in a vacuum depends only on the relative velocity between the emitter and the receiver is resolved by using the Lorentz transformations instead of the Galilean ones. As a direct consequence, the well known relativistic aberration effect was also derived Eq. (35).

It is our hope that courses in special relativity will be structured to introduce relativity theory through the Doppler effect in the way that is outlined in this paper.

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APPENDIX A: NON-RELATIVISTIC ABERRATION OF WAVES DUE THE MOTION OF THE RECEIVER

Angular aberration is the effect that the angle of arrival depends on the relative velocity between the emitter and the receiver. This effect is well known and was one of the first ways in which the speed of light was measured.[17]

The invariance of the wave vector $\boldsymbol{k}$ under Galilean transformations would seem to suggest that the angle of arrival of a plane wave is independent of the velocity of the receiver or emitter, in contradiction with aberration theory. As discussed in Ref. 18, the angle of arrival of a plane wave is determined by its group velocity, not its phase velocity. It is apparent from Eq. (16) that angular aberration occurs when either the receiver or the emitter is moving, even though the wave vector $\boldsymbol{k}$ is unchanged.

To calculate the aberration effect, consider the angle between the x-axis and the group velocity $\boldsymbol{u}_g$ for two observers: $\mathcal{O}$, who measures the group velocity as $\boldsymbol{u}_g$, and $\mathcal{O}'$, who measures the group velocity as $\boldsymbol{u}'_g$ and is moving with a velocity $\boldsymbol{v}$ in the s-direction with respect to $\mathcal{O}$, i.e.,

$$\boldsymbol{v} = v\hat{\boldsymbol{x}}, \tag{A1}$$

as shown in Fig. 9. In this scenario for observer $\mathcal{O}$ the angle between the group velocity and the x-axis is

$$\cos\theta = \frac{\boldsymbol{u}_g^T\hat{\boldsymbol{x}}}{u_g}. \tag{A2}$$

Similarly, according to $\mathcal{O}'$ this angle is

$$\cos\theta' = \frac{\boldsymbol{u}_g'^T\hat{\boldsymbol{x}}'}{u'_g}. \tag{A3}$$

We can determine the relationship between θ and θ' by noting that, by construction, the $\mathcal{O}$ and $\mathcal{O}'$ axes are aligned, so that $\hat{\boldsymbol{x}}' = \hat{\boldsymbol{x}}$. Furthermore, in Sec. II D, we showed that if $\mathcal{O}$ measures the group velocity as $\boldsymbol{u}_g$, then according to $\mathcal{O}'$ it is $\boldsymbol{u}'_g = \boldsymbol{u}_g - \boldsymbol{v}$, and hence

$$\cos\theta' = \frac{\boldsymbol{u}_g'^T\hat{\boldsymbol{x}}}{u'_g} = \frac{\boldsymbol{u}_g^T\hat{\boldsymbol{x}} - v}{u'_g} = \frac{u_g\cos\theta - v}{u'_g}. \tag{A4}$$

If the group speed is much greater than v, then $v/u_g \ll 1$ and to first order in v/u_g we have

$$u'_g \approx u_g - \frac{\boldsymbol{u}_g^T\boldsymbol{v}}{u_g} = u_g - v\cos\theta, \tag{A5}$$

from which we conclude that

$$\begin{aligned}\cos\theta' &\approx \frac{u_g\cos\theta - v}{u_g - v\cos\theta} \approx \cos\theta + \frac{v}{u_g}\cos^2\theta - \frac{v}{u_g}\\ &= \cos\theta - \frac{v}{u_g}\sin^2\theta.\end{aligned} \tag{A6}$$

Finally we note that if the aberration angle is small so that $\theta' = \theta + \Delta\theta$, then

$$\Delta\theta \approx \frac{v}{u_g}\sin\theta. \tag{A7}$$

In the case of light $u_g = c$ and this aberration is known as stellar aberration or Bradley aberration, as it was used to determine the speed of light by Bradley.[17]

APPENDIX B: THE DOPPLER EFFECT FROM THE LORENTZ TRANSFORMATIONS

In the rest frame of the medium, the receiver is moving with a velocity $\boldsymbol{v}_r$, so from Eq. (31) we get

$$f_r = f_m\left(1 - \frac{\boldsymbol{v}_r^T\boldsymbol{u}}{u_p^2}\right)\gamma_r, \tag{B1}$$

where $\gamma_r = 1/\sqrt{1 - v_r^2/c^2}$. Similarly, in the rest frame of the medium the emitter is moving with a velocity $\boldsymbol{v}_e$, and the relativistic Doppler effect is

$$f_e = f_m \left(1 - \frac{\boldsymbol{v}_e^T \boldsymbol{u}}{u_p^2} \right) \gamma_e, \tag{B2}$$

where $\gamma_e = 1/\sqrt{1 - v_e^2/c^2}$. Eliminating f_m from these equations we obtain Eq. (32):

$$f_r = \frac{\sqrt{1 - v_e^2/c^2}}{\sqrt{1 - v_r^2/c^2}} \frac{\left(u_p^2 - \boldsymbol{v}_r^T \boldsymbol{u}\right)}{\left(u_p^2 - \boldsymbol{v}_e^T \boldsymbol{u}\right)} f_e. \tag{B3}$$

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[11]One aspect of our notation that is not standard is that we write the dot product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ in linear algebra notation: $\boldsymbol{a}^T \boldsymbol{b}$, not in the more common $\boldsymbol{a} \cdot \boldsymbol{b}$ notation. The main reason for this is that as students advance in their studies they will be asked to calculate dot products in general space-times, which are defined by a metric tensor $\boldsymbol{G}$, for which the dot product is

$$(\boldsymbol{a} \cdot \boldsymbol{b})_{\mathrm{G}} \equiv \boldsymbol{a}^T \boldsymbol{G} \boldsymbol{b}. \tag{B4}$$

The Euclidean dot product $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{a}^T \boldsymbol{b}$ is a special case in which the metric tensor is the identity matrix ($\boldsymbol{G} = \boldsymbol{I}$). Another advantage of the linear algebra notation is that it translates more easily into programming languages. For example, if `a` and `b` are $D \times 1$ column vectors and `G` is a $D \times D$ matrix, then the Matlab code that generates the numerical value of Eq. (B4) is `a'*G*b`, where ' is the transpose operation in Matlab.

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[14]The group velocity equation (15) is normally written as $\boldsymbol{u}_g = 2\pi \nabla_{\boldsymbol{k}} f$, where $\nabla_{\boldsymbol{k}} \equiv [\partial/\partial k^x, \partial/\partial k^y, \partial/\partial k^z]$. However, this notation only makes sense if Eq. (11) is written as $f' = f - \boldsymbol{k} \cdot \boldsymbol{v}$, where $\cdot$ denotes the usual dot product. In linear algebra notation $\boldsymbol{k} \cdot \boldsymbol{v}$ is expressed as $\boldsymbol{k}^T \boldsymbol{v}$ and hence Eq. (15) is the correct expression for the group velocity.

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