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POST-NEWTONIAN n-BODY EQUATIONS OF THE BRANS-DICKE THEORY*

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ABSTRACT

The post-Newtonian equations of the Brans-Dicke scalar-tensor theory are derived, for the case of n gravitating point masses. They are a set of coupled second-order differential equations for the accelerations of the point masses, which prove to be derivable from a classical velocity-dependent Lagrangian.

Chandrasekhar (1965) and Nutku (1969) have derived post-Newtonian equations of hydrodynamics in the scalar-tensor theory of Brans and Dicke (1961). Nutku obtains the metric coefficients in this approximation, the equations of motion for a nonviscous perfect fluid, the conservation of proper mass, and the integrals of linear momentum, angular momentum, and energy. This has made possible a quick derivation of the equations of motion of n gravitating point masses, as well as the discovery of an exact classical Lagrangian, for this case. It is expected that these equations will be useful in analysis of data from ultraprecise radar tracking of space probes and planets.

In the Newtonian approximation, the metric in the Brans-Dicke theory is, in isotropic coordinates,

$$\begin{aligned} g_{00} &= 1 - \frac{2U}{c^2} + O(c^{-4}) \,, \\ g_{0a} &= O(c^{-3}) \,, \\ g_{ab} &= -\delta_{ab} - \frac{1+\omega}{2+\omega}\frac{2U}{c^2}\delta_{ab} + O(c^{-4}) \,, \end{aligned} \tag{1}$$

where U is the Newtonian potential, ω is the coupling constant of the scalar field ($\omega \rightarrow \infty$ is the case of Einsteinian general relativity), and $a, b = 1, 2, 3$. These expressions suffice for discussion of the propagation of light in the solar system.

The post-Newtonian terms in this slow-motion expansion of the metric are obtained with the metric field equations and the wave equation for the scalar field; they are necessary for discussion of relativistic terms in the motion of planets and in space navigation. We specialize Nutku's results to the n-body case by recognizing that his conserved mass density ρ^*, which is equal to $\rho[1 + (v^2/2c^2) + Uc^{-2}(3 + 3\omega)/(2 + \omega)]$, where v is the fluid speed, is, from equation (1), the (relative) density of proper mass. (We ignore his, and Chandrasekhar's [1965], distinction between total internal energy density ϵ and material energy density ρc^2.) We rewrite his expressions for the metric coefficients in terms of volume integrals of ρ^*, and then renormalize the proper mass density to include the gravitating pressure stresses; i.e., we replace in his integrals $[\rho^* + 3pc^{-2}(1 + \omega)/(2 + \omega)]dV$ by just dm, and then write a finite sum over discrete masses m_j; (cf. the notes of H. P. Robertson [Robertson and Noonan 1968]). Our model of a planet is thus a spherically symmetric, isentropic, stationary, self-gravitating mass of perfect fluid.

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The resulting post-Newtonian n-body metric is

$$
\begin{aligned}
g_{00} &= 1 - \frac{2U}{c^2} + \frac{2U^2}{c^4} - \frac{4+3\omega}{2+\omega}\frac{1}{c^4}\sum_j \frac{\mu_j v_j^2}{r_j} + \frac{2}{c^4}\sum_j \frac{\mu_j}{r_j}\sum_{k\neq j}\frac{\mu_k}{r_{jk}} + \frac{1}{c^4}\frac{\partial^2\chi}{\partial t^2}\,, \\
g_{0a} &= +\frac{1}{c^3}\frac{6+4\omega}{2+\omega}\sum_j \frac{\mu_j}{r_j}\frac{d\boldsymbol{r}_j}{dt}\,, \qquad (2)\\
g_{ab} &= -\left(1 + \frac{2+2\omega}{2+\omega}\frac{1}{c^2}U\right)\delta_{ab}\,.
\end{aligned}
$$

In equation (2) $\boldsymbol{r}$ locates the field point at which $g_{\mu\nu}$ is given; $\boldsymbol{r}_j$ locates the jth particle, with gravitating mass $\mu_j = G_0 m_j$, G_0 being the Newtonian constant of gravitation. However, the scalar $r_j = |\boldsymbol{r}_j - \boldsymbol{r}|$, and $r_{jk} = |\boldsymbol{r}_j - \boldsymbol{r}_k|$. The vector components of $d\boldsymbol{r}_j/dt$ are understood in the expression for g_{0a}. $U = \Sigma_j \mu_j/r_j$ as before. The quantity $\chi = -\Sigma_j \mu_j r_j$ is denoted the superpotential by Chandrasekhar (1965); in equation (2) the point coordinates of the field are to be held fixed in forming the second partial of χ. We have also found it convenient to perform a gauge transformation on the metric of Nutku. The metric (2) satisfies a radiation coordinate condition (Robertson and Noonan 1968).

The equations of motion follow in standard fashion; each of the particles is self-consistently and successively treated as a *test* particle moving on a geodesic of a non-singular metric obtained from equation (2) by suppressing its own direct contributions to the summations. Furthermore, in the differentiations involved, care must be taken to regard the potentials and accelerations of the *other* masses as given.

The result is, for the ith particle,

$$
\begin{aligned}
\frac{d^2\boldsymbol{r}_i}{dt^2} = \sum_{j\neq i}\frac{\mu_j(\boldsymbol{r}_j - \boldsymbol{r}_i)}{r_{ij}^3}\bigg\{ 1 &- \frac{1}{c^2}\frac{6+4\omega}{2+\omega}\sum_{l\neq i}\frac{\mu_l}{r_{il}} - \frac{1}{c^2}\sum_{k\neq j}\frac{\mu_k}{r_{jk}} \\
&+ \frac{1}{c^2}\frac{1+\omega}{2+\omega}v_i^2 + \frac{1}{c^2}\frac{3+2\omega}{2+\omega}v_j^2 - \frac{2}{c^2}\frac{3+2\omega}{2+\omega}\frac{d\boldsymbol{r}_i}{dt}\cdot\frac{d\boldsymbol{r}_j}{dt} \\
&- \frac{3}{2c^2}\left[\frac{1}{r_{ij}}(\boldsymbol{r}_j - \boldsymbol{r}_i)\cdot d\boldsymbol{r}_j/dt\right]^2 + \frac{1}{2c^2}(\boldsymbol{r}_j - \boldsymbol{r}_i)\cdot\frac{d^2\boldsymbol{r}_j}{dt^2}\bigg\} \qquad (3)\\
&- \frac{1}{c^2}\sum_{j\neq i}\frac{\mu_j}{r_{ij}^3}(\boldsymbol{r}_j - \boldsymbol{r}_i)\cdot\left(\frac{4+3\omega}{2+\omega}\frac{d\boldsymbol{r}_j}{dt} - \frac{6+4\omega}{2+\omega}\frac{d\boldsymbol{r}_i}{dt}\right)\left(\frac{d\boldsymbol{r}_j}{dt} - \frac{d\boldsymbol{r}_i}{dt}\right) \\
&+ \frac{1}{2c^2}\frac{10+7\omega}{2+\omega}\sum_{j\neq i}\frac{\mu_j}{r_{ij}}\frac{d^2\boldsymbol{r}_j}{dt^2}\,,
\end{aligned}
$$

where $v_i^2 = |d\boldsymbol{r}_i/dt|^2$. In the last terms, we may substitute the Newtonian value

$$
d^2\boldsymbol{r}_j/dt^2 = \sum_{k\neq j}\frac{\mu_k(\boldsymbol{r}_k - \boldsymbol{r}_j)}{r_{jk}^3}\,.
$$

For the case of one body orbiting a massive second one (mass M), these equations give an advance of perihelion $(6M^2G_0^2/h^2c^2)\,(4+3\omega)/(6+3\omega)\,\pi$ radians per revolution, as already known (Brans and Dicke 1961).

It is remarkable, although not unsuspected because of the elegant integral theorems of Nutku, that equation (3) is derivable from a classical Lagrangian. We find

$$L = \sum_j \frac{\mu_j v_j^2}{2}\left(1 + \frac{4+3\omega}{2+\omega}\sum_{k\neq j}\frac{\mu_k}{c^2 r_{jk}}\right) + \sum_j \frac{\mu_j v_j^4}{8c^2} + \sum_j \sum_{k\neq j}\frac{\mu_j \mu_k}{2r_{jk}}$$
$$- \sum_j \sum_{k\neq j}\frac{\mu_j \mu_k}{4c^2 r_{jk}}\left[\frac{10+7\omega}{2+\omega}\frac{dr_j}{dt}\cdot\frac{dr_k}{dt} + \left(\frac{dr_j}{dt}\cdot n_{jk}\right)\left(\frac{dr_k}{dt}\cdot n_{jk}\right)\right] \quad (4)$$
$$- \sum_j \sum_{k\neq j}\sum_{l\neq j}\frac{\mu_j \mu_k \mu_l}{2c^2 r_{jk} r_{jl}} .$$

Here $n_{jk} = (r_j - r_k)/r_{jk}$. This reduces immediately to the result given in Landau and Lifshitz (1962). It is interesting that the resulting conserved mass and linear momentum are independent of ω.

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